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Matrix treatment of wave propagation in stratified media

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Abstract. An analytic approximation for the diffeomorphism of a homogeneous linear second-order differential equation is obtained in matrix representation. As a consequence of energy conservation for waves propagating in a non-absorbing stratified medium the corresponding transmission matrix belongs to the group QU(2). The approximation contains the WKB approximation and may be applied to discrete and continuous media. As an application of the method three specific problems are treated: calculation of the reflection coefficient, determination of the eigenmodes, and calculation of the adiabatic invariant for a damped classical harmonic oscillator.

1. Introduction

The propagation of plane monochromatic waves in stratified inhomogeneous media is governed by the one-dimensional form of the amplitude equation (Landau and Lifshitz 1967),

$$u''-(\nu'/\nu)u'+k^2u=0, \tag{1}$$

where $k(x)$ is the x component of the wavevector, i.e. the component normal to the strata. In the case of electromagnetic waves, the two polarizations must be distinguished. If the electric field is perpendicular to the plane of incidence, $u(x)$ denotes the complex amplitude of that field and $\nu'=0$; for formal reasons we are going to use $\nu=1$. In the other case u is the amplitude of the magnetic field and $\nu(x)$ equals the dielectric constant.

An equation of the same form, with x representing the time describes the motion of a damped harmonic oscillator with varying frequency $k(x)$ and varying damping parameter. (In the usual notation the latter equals $-\nu'/(2\nu)$.) The frequency may vary due to adiabatic changes or, in another extreme, due to parametric excitation (Landau and Lifshitz 1960, Arnol'd 1974, Jordan and Smith 1977).

The subsequent analysis is intended to derive an analytic approximation for the diffeomorphism (Nitecki 1971) corresponding to equation (1) and valid for not too large and not too fast changes of the parameters k and ν. The diffeomorphism which relates $u(x)$ to the initial value $u(x_0)$, will be represented by a transmission matrix $G(x_0, x)$. Once G is known, the general solution of equation (1) is completely determined.

The transmission matrix appears to be the fundamental tool of the present treatment of wave propagation in stratified media. With use of that matrix the formalism becomes simple and compact, which seems to be an advantage of the method. Indeed, in §§ 7–9 it will be shown how easily some specific results can be derived. The underlying approximation will be seen to be more accurate than the first-order WKB approximation, so that results in §§ 7–9 appear as straightforward generalisations of the corresponding WKB results. For these reasons the present matrix method might be regarded, in certain respects, as more effective than the WKB method.

2. Matrix representation of the diffeomorphism

The function $u(x)$ will be assumed to take complex values, while the parameter ν together with k shall be real and positive, which in the case of wave propagation excludes absorption.

Let the solution $u(x)$ be decomposed into travelling waves,

$$u(x)=\left(\frac{\nu(x)}{k(x)}\right)^{1/2}(A(x)\,\mathrm{e}^{\mathrm{i}kx}+B(x)\,\mathrm{e}^{-\mathrm{i}kx}). \tag{2}$$

If complex amplitudes $A(x)$ and $B(x)$ are permitted, the splitting is not unique. It will be specified by the approximation worked out in §§ 4 and 5 and by the boundary conditions. For instance if the inhomogeneities are confined to the interval (x_0, x_1), and we only have an incident wave from the left, its amplitude shall equal $A(x)$ for $x<x_0$, while $B(x)=0$ for $x>x_1$.

It will be convenient to introduce a two-dimensional complex-amplitude vector $v=(A, B)$. We look for the transmission matrix $G(x_0, x)$ which transforms $v(x_0)=v_0$ into $v(x)$, i.e.

$$v(x)=G(x_0, x)v_0. \tag{3}$$

Obviously, one has to require $G(x_0, x_0)=I$, where I is the identity matrix. With the vector $f=(\nu/k)^{1/2}(\mathrm{e}^{-\mathrm{i}kx}, \mathrm{e}^{\mathrm{i}kx})$ and the usual notation for the inner product the solution u appears in compact form,

$$u=\langle v, f\rangle=\langle Gv_0, f\rangle. \tag{4}$$

3. Group property of transmission matrices

In the spirit of the geometrical-optics approximation we assume that changes of ν and k over a few periods of u can be neglected. Then the net energy flow in the x direction averaged over a period is found to be proportional to the quasinorm $\|v\|=|A|^2-|B|^2$ of the amplitude vector. Formally, this can be written as an inner product $\|v\|=\langle v, \sigma_z v\rangle$ with the aid of one of the Pauli matrices,

$$\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_y=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \qquad \sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Conservation of energy requires the quasinorm of the amplitude vector to remain constant as x varies, hence

$$\langle Gv_0, \sigma_z Gv_0\rangle=\langle v_0, \sigma_z v_0\rangle. \tag{9}$$

This is equivalent to

$$G^{\dagger}\sigma_z G=\sigma_z,$$

where $G^{\dagger}$ is the adjoint of G. A general representation of G in terms of three real parameters easily follows (Vilenkin 1965)

$$G = \exp(\mathrm{i}\phi\sigma_z)\exp(\tau\sigma_x)\exp(\mathrm{i}\psi\sigma_z), \tag{6}$$

or explicitly,

$$G = \begin{bmatrix} e^{\mathrm{i}(\phi+\psi)}\cosh\tau & e^{\mathrm{i}(\phi-\psi)}\sinh\tau \\ e^{-\mathrm{i}(\phi-\psi)}\sinh\tau & e^{-\mathrm{i}(\phi+\psi)}\cosh\tau \end{bmatrix}. \tag{7}$$

The parameter τ may be arbitrary, while ϕ and ψ are confined to the interval $[0, 2\pi)$ if uniqueness of the representation is requested. Since $G(x_0, x_0) = I$, the determinant of the transmission matrices must be 1. All matrices of this type form the group QU(2) (Vilenkin 1965). We also notice that $G_{11} = G_{22}^{*}$ and $G_{12} = G_{21}^{*}$. (However, G is not self-adjoint except when $\phi = -\psi$.) This symmetry ensures that u remains real if its value was real initially. Such a situation describes a standing wave where the net energy flow vanishes.

4. Differential equation for the transmission matrix

Substitution of expression (4) into equation (1) results in

$$\langle[G'' - (\nu'/\nu)G' + k^2 G]v_0, f\rangle + \langle[2G' - (\nu'/\nu)G]v_0, f\rangle + \langle Gv_0, f''\rangle = 0.$$

Since f' is linearly related to f,

$$f' = -E^{\dagger} f, \qquad E = \frac{1}{2}\left(\frac{k'}{k} - \frac{\nu'}{\nu}\right)I - \mathrm{i}(k'x + k)\sigma_z, \tag{8}$$

and because v_0 is arbitrary, a differential equation for G itself follows,

$$[G'' - (\nu'/\nu)G' + k^2 G] - E[2G - (\nu'/\nu)G] + (E^2 - E')G = 0.$$

The substitution

$$G' = HG \tag{9}$$

leads to a Riccati equation for the matrix H:

$$H' - E' + (H - E)H - E(H - E) - (\nu'/\nu)(H - E) + k^2 I = 0. \tag{10}$$

We shall attempt an approximate solution of this equation and then determine G from equation (9).

It is interesting to note that the approximation $H = E$ satisfies this equation up to terms of second order in k, and thus may be used for small frequencies. However, geometrical optics no longer applies and the resulting transmission matrices do not belong to the group QU(2). Moreover, the previously imposed conditions on the parameters ν and k become irrelevant.

A consistent WKB approximation for H can hardly be derived from equation (10) directly. We therefore proceed by a different approach and subsequently verify how well equation (10) is obeyed.

5. Step-function approximation

Now we imagine a continuously varying parameter k as approximated by a piecewise constant function with sufficiently small jumps. Then, in a neighbourhood around a single jump placed in the interval $(x, x+\Delta x)$ we neglect the derivatives of k. The Fresnel boundary conditions entering the calculation are that u and $\nu^{-1}u'$ are continuous at the point of discontinuity of k. (The parameter ν may also be discontinuous at the same point.) A relation

$$v_+ = \Delta G(\Delta x) v_- \tag{11}$$

between the amplitude vectors v_+ and v_- to the right and to the left, respectively, from the jump follows. The corresponding parameters of the matrix $\Delta G(\Delta x)$ from QU(2), according to equation (6), are

$$\phi = -k_+ x, \qquad \tau = \frac{1}{2}\ln\left(\frac{k_+\nu_-}{k_-\nu_+}\right), \qquad \psi = k_- x, \tag{12}$$

if the discontinuity is at x. With this result we can write

$$G(x_0, x+\Delta x) = \Delta G(\Delta x) G(x_0, x),$$

which still refers to the step function. However, in the limit $\Delta x \to 0$ we hope to obtain

$$G' = \lim_{\Delta x \to 0}[(\Delta x)^{-1}(G(x_0, x+\Delta x) - G(x_0, x))],$$

corresponding now to the original function $k(x)$. (At this step we have locally neglected the derivatives of k, which will become apparent by the calculation of the error (14) below.)

Equation (9) then takes the form

$$H = \lim_{\Delta x \to 0}[(\Delta x)^{-1}(\Delta G(\Delta x) - I)].$$

With the parameters ϕ, τ, ψ from (12) and a series expansion of the expressions in the intermediate result an explicit approximation follows:

$$H = -\mathrm{i}k'x\sigma_z + \frac{1}{2}\left(\frac{k'}{k} - \frac{\nu'}{\nu}\right)[\sigma_x \cos(2kx) + \sigma_y \sin(2kx)]. \tag{13}$$

We verify that equation (10) is satisfied up to the error

$$\frac{1}{2}\left[2\left(\frac{k'}{k}\right)^2 - \frac{k''}{k} - \frac{\nu'k'}{\nu k} - \left(\frac{\nu'}{\nu}\right)^2 + \frac{\nu''}{\nu}\right][I - \sigma_x \cos(2kx) - \sigma_y \sin(2kx)]. \tag{14}$$

Obviously, the factor in front is small for smoothly varying k and ν. Tc assess the accuracy of the approximation (13), which will be used in equation (9), we take ν = constant and compare the first term of the error (14) with the last term of equation (10). We conclude that the approximation is justified if $|k'/k^2|^2 \ll 1$. On the other hand, the first-order WKB approximation, which is frequently used, is satisfactory provided $|k'/k^2| \ll 1$. Although a complete and explicit comparison with the WKB method is lacking, we may state that the established approximation leads to results whose accuracy is comparable with the accuracy of the higher-order WKB results. Specific results obtained in §§ 7–8 confirm our belief.

6. Approximate solution for the transmission matrix

Since the matrices H and G do not commute, the solution of $G' = HG$ with H from equation (13) is not trivial. The substitution

$$G = \exp\left(-i\sigma_z \int_{x_0}^{x} yk'(y)\,dy\right) N \tag{15}$$

facilities the task. The matrix N is seen to obey the equation

$$N' = MN, \qquad M = \frac{1}{2}\left(\frac{k'}{k} - \frac{\nu'}{\nu}\right)[\sigma_x \cos(2\beta) + \sigma_y \sin(2\beta)], \tag{16}$$

where $\beta = k_0 x_0 + \int_{x_0}^{x} k(y)\,dy$ is the phase. If the commutator $[M, \int_{x_0}^{x} M(y)\,dy]$ vanishes, the exact solution of the last equation can be immediately given as $\exp(\int_{x_0}^{x} M(y)\,dy)$. However, the commutator is generally equal to $i\delta\sigma_z$, where

$$\delta = 2\,\mathrm{Im}\left[s\left(\frac{ds}{dx}\right)^*\right], \tag{17}$$

$$s = \int_{x_0}^{x} \frac{1}{2}\left(\frac{k'}{k} - \frac{\nu'}{\nu}\right) e^{2i\beta}\,dy. \tag{18}$$

It is apparent that δ is the relevant small parameter which determines the accuracy of the approximate solution.

As already noted, for $\delta \approx 0$ we obtain the following approximation for the transmission matrix:

$$G(x_0, x) = \exp\left(-i\sigma_z \int_{x_0}^{x} yk'(y)\,dy\right) \exp\left(\int_{x_0}^{x} M(y)\,dy\right), \tag{19}$$

which turns out to be satisfactory in many cases. Looking at the expression (6) and doing some manipulations with the series expansion we infer the values of the parameters

$$\phi = -\left(\tfrac{1}{2}\arg s + \int_{x_0}^{x} yk'(y)\,dy\right), \qquad \tau = |s|, \qquad \psi = \tfrac{1}{2}\arg s, \tag{20}$$

with s specified in equation (18).

Exactly soluble examples of equation (1) show that the stated approximation for G is close to exact solution even for discontinuously varying $k(x)$. The reason is that δ is proportional to the derivative of s, so that the error accumulates relative slowly. Of course, for a single step of $k(x)$ the result is exact by the nature of our derivation in § 5.

7. Reflection coefficient for electromagnetic waves

We are interested in determining the reflection coefficient r for a given layer (x_0, x_1), surrounded with homogeneous media on both sides. The definition is, according to equation (2), $r = e^{-2ik_0x_0} B(x_0)/A(x_0)$. Also the boundary condition $B(x) = 0$ for $x > x_1$ must be taken into account. For the two polarisations r refers to the electric and magnetic field amplitudes, respectively. The reflectance is equal to $|r|^2$.

Since $B(x_0)$ is unknown, we have to calculate $G(x_0, x_1)$ in order to relate $v_0 = (A(x_0), B(x_0))$ to $V_1 = (A(x_1), 0)$. Once s from equation (18) is calculated, the parameters τ and ψ from (20) entering the expression (7) are specified. Then the final result follows as

$$r = -e^{-2ik_0x_0} e^{2i\psi} \tanh \tau = -e^{-2ik_0x_0}(s \tanh |s|)/|s|. \tag{21}$$

Neither ϕ nor ψ are needed to express the reflectance which is determined solely by the parameter $\tau = |s|$. Namely, $|r|^2 = \tanh^2 |s|$.

From equation (17) we see that the approximation (21) is justified for small $|s|$, which corresponds to the WKB approximation, and for all cases where the reflectance is extremal, because here the derivative of s vanishes. Actually, for small $|s|$ we obtain

$$r \cong -\int_{x_0}^{x_1} \frac{1}{2}\left(\frac{k'}{k} - \frac{\nu'}{\nu}\right) \exp\left(2i \int_{x_0}^{x} k(y)\, dy\right) dx, \tag{22}$$

which agrees with the WKB approximation (Babikov 1976). More generally, the result (21) is in agreement with the formula derived by Greenewalt *et al* (1960) which was established by a less general approach based on the Ricatti differential equation for the reflection coefficient itself (for TE polarisation only).

8. Eigenmodes of the field

In this section we attack the eigenvalue problem corresponding to the wave equation (1) with the boundary conditions that the field u vanishes at the boundaries x_0 and x_1. We may think of the electromagnetic field in a stratified dielectric between two perfectly conducting plates, or, e.g., of a vibrating string fixed at the end points x_0 and x_1. Our aim is to determine the eigenmodes.

First, one should notice that v_0 need not be zero if u_0 vanishes. Hence, in spite of the boundary conditions the amplitude vectors v_0 and $v_1 = G(x_0, x_1)v_0$ do not vanish in general. So we choose an arbitrary $v_0 = (A_0, B_0)$ and require the boundary conditions

$$u_0 = \langle v_0, f_0\rangle = 0, \qquad u_1 = \langle Gv_0, f_1\rangle = \langle v_0, G^{\dagger} f_1\rangle = 0, \tag{23}$$

to be fulfilled. This system of equations for the unknowns A_0 and B_0 has a non-trivial solution (which implies that $u(x) = \langle G(x_0, x)v_0, f(x)\rangle$ is non-trivial) if the determinant vanishes, which means

$$\operatorname{Im}\{[G_{11} \exp(-ik_0x_0) - G_{12} \exp(ik_0x_0)] \exp(ik_1x_1)\} = 0.$$

With equations (7) and (20) this condition determines the eigenfrequencies, and is equivalent to

$$\operatorname{Im}\left[(1 + r(x_0, x_1)) \exp\left(-i \int_{x_0}^{x_1} k(y)\, dy\right)\right] = 0, \tag{24}$$

where by $r(x_0, x_1)$ the reflection coefficient corresponding to the layer (x_0, x_1) is denoted (see equation (21)). The present method apparently leads to the generalisation (24) of the well known WKB result $\int_{x_0}^{x_1} k(y)\, dy = n\pi$, $n = 1, 2, \ldots$, which is correct only if the corresponding reflection coefficient r is negligible.

The initial amplitude vector v_0 arising from equation (23) together with the transmission matrices then determines the eigenmodes, according to equation (4). These, of course, must be standing waves, so that $B_0 = A_0^*$ might be required in addition to the boundary conditions (23).

The derivation is essentially the same if other boundary conditions are used. For instance, if $u'(x_0)=0$ and $u(x_1)=0$, we get

$$\mathrm{Re}\left[(1+r(x_0,x_1))\exp\left(-\mathrm{i}\int_{x_0}^{x_1} k(y)\,\mathrm{d}y\right)\right]=0.$$

9. Adiabatic invariant of the harmonic oscillator

In order to describe the motion of a damped classical harmonic oscillator with continuously varying frequency k and damping parameter $\Lambda=-\nu'/2\nu$ we require the solution u to be real for all values of time x. It suffices to state that $v_0=(A_0, A_0^*)$, as the required condition is then automatically satisfied for each x as was pointed in § 3. The energy of the oscillator averaged over the period of u is proportional to kJ, where $J=\nu|A|^2$ is the adiabatic invariant (Landau and Lifshitz 1960, Arnol'd 1974). If expression (7) with the parameters from (20) is substituted into $v(x)=G(x_0,x)v_0$, we have

$$J=(\nu/\nu_0)[\cosh(2|s|)+\sinh(2|s|)\cos(\arg s+2\arg A_0)]J_0. \tag{25}$$

The parameter s is taken from equation (18), while J_0 and ν_0 refer to the initial values. The definition $\Lambda=-\nu'/2\nu$ can be integrated in order to express ν with the damping parameter Λ. According to equation (25), the result

$$\nu=\nu_0\exp\left(-2\int_{x_0}^{x}\Lambda(y)\,\mathrm{d}y\right) \tag{26}$$

describes the decay of energy if the frequency k is constant, and, in addition, if ν varies slowly in comparison with $e^{2i\beta}$ so that s from equation (18) can be neglected. The general case, where J varies due to damping and due to adiabatic changes of the parameters k and ν, is then described by equation (25). In absence of damping and in the limit $|s|\to 0$ the adiabatic invariant is seen to be conserved, so that energy varies in proportion to frequency (Landau and Lifshitz 1960).

If the frequency is discontinuous, the formalism must be applied with care. In addition to the continuity of u we have the continuity of the velocity u', but not necessarily of $\nu^{-1}u'$. Thus, if ν and k are discontinuous simultaneously, the preceding results can not be correct.

The second term in the brackets of equation (25) explains the dependence of J upon the initial phase. This information could be used in the case of weak parametric excitation whose period is much longer than the period of u. Suppose that ν is continuous and k changes discretely in such a manner that the argument of the cosine is zero or π all the time. We see then that J is exponentially increasing or decreasing, respectively. This is consistent with the behaviour of u under parameter excitation (Landau and Lifshitz 1960, Arnol'd 1974, Jordan and Smith 1977).

10. Comments

Perhaps the present matrix method of solving the one-dimensional wave equation (1) could be improved if different discrete approximations, e.g. with piecewise linear functions instead of step functions, were used. Beside this extension of the accuracy another important generalisation is desirable. Namely, the restriction that k and ν are real should be removed so that one could treat absorbing media, and the bound states of quantum mechanical systems.

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References

Arnol'd V I 1974 *Mathematical Methods of Classical Mechanics* (in Russian) (Moscow: Nauka) § 25
Babikov V V 1976 *Phase-Functions Method in Quantum Mechanics* (in Russian) (Moscow: Nauka) § 18
Grenewalt C H W, Brandt W and Friel D D 1960 *J. Opt. Soc. Am.* **50** 1005–13
Jordan D W and Smith P 1977 *Nonlinear Ordinary Differential Equations* (Oxford: Clarendon) p 238
Landau L D and Lifshitz E M 1960 *Mechanics* (Massachusetts: Addison-Wesley) § 27, § 49
—— 1967 *Elektrodynamik der Kontinua* (Berlin: Akademie-Verlag) § 68
Nitecki Z 1971 *Differentiable Dynamics* (Cambridge, Mass.: MIT Press)
Vilenkin N Ya 1965 *Special Functions and the Theory of Group Representations* (in Russian) (Moscow: Nauka) chap. 6